(4.4) a) Since \( wt \) is white noise, \( E[w_t] = 0 \) and
\[
\delta_w(h) = \begin{cases} 
1, & h = 0 \\
0, & \text{otherwise.}
\end{cases}
\]
Since \( E[w_t] \) is free of \( t \) and the ACVF depends only on \( h \), \( wt \) is stationary.
\[
E[X_t] = E[w_t] = 0, \quad \text{and} \quad \gamma(h, t+h) = \text{cov}(X_t, X_{t+h})
\]
\[
= \text{cov}(w_t, w_{t+h} - \Theta w_{t+h-1})
\]
\[
= \begin{cases} 
1 + \Theta^2, & h = 0 \\
-\Theta, & h = \pm 1 \\
0, & \text{otherwise.}
\end{cases}
\]
Since \( E[X_t] \) is free of \( t \) and \( \gamma(h, t+h) \) depends only on \( h \), \( X_t \) is stationary.

(continued)
(continued)

\[
\begin{align*}
\theta(w) & = \sum_{h=-\infty}^{\infty} \delta(h) e^{-2\pi i wh} \\
& = \sum_{h=-1}^{1} \delta(h) e^{-2\pi i wh} \\
& = (1 + \theta^2) - \theta \left[ e^{-2\pi i w} + e^{-2\pi i \omega(-1)} \right] \\
& < 1 + \theta^2 - \theta \left[ \cos(-2\pi w) + i\sin(-2\pi w) + \cos(2\pi w) + i\sin(2\pi w) \right] \\
& = \left[ 1 + \theta^2 - 2\theta \cos(2\pi w) \right] \quad \text{(since \(\cos\) is an even function and \(\sin\) is an odd function)}
\end{align*}
\]
The method from above is to check that the corresponding spectral density is never negative. Here the spectral density is

\[ \gamma(w) = \sum_{k=0}^{\infty} \gamma(k) e^{-2\pi iwh} \quad (\text{for } -\frac{1}{2} \leq w \leq \frac{1}{2}) \]

\[ = 1 \cdot e^{0} - 0.5 e^{-2\pi i(-2)} = 1 - 0.5 e^{-2\pi i(-2)} \]

\[ - 0.5 e^{-2\pi i(-2)} - 0.25 e^{-2\pi i(-3)} - 0.25 e^{-2\pi i(-3)} = 1 - 0.5 \cos(4\pi) - 0.5 \cos(6\pi), \]

If \( w = 0 \), this is \( 1 - 1 - 0.5 = -0.5 < 0 \).

Thus, the function \( \gamma(w) \) cannot be the ACF for a stationary process.

Let \( f_x(w) \) and \( f_y(w) \) be the spectral densities for \( X_t \) and \( Y_t \). Then the spectral density for \( X_t + Y_t \) is

\[ f_{X+Y}(w) = \sum_{k=-\infty}^{\infty} \gamma_{xy}(k) e^{-2\pi iwh} \]

Since \( X_t \) and \( Y_t \) are uncorrelated, \( \gamma_{xy}(k) = \gamma_x(k) + \gamma_y(k) \).
(3) 
(continued)

Thus, 
\[ f_{x+y}(w) = \sum_{h=-\infty}^{\infty} \delta_{x+y}(h) e^{-2\pi i wh} \]
\[ = \sum_{h=-\infty}^{\infty} \delta_{x}(h) e^{-2\pi i wh} + \sum_{h=-\infty}^{\infty} \delta_{y}(h) e^{-2\pi i wh} \]
\[ = f_{x}(w) + f_{y}(w). \]

(4) We first need the ACVF. We assume that

\[ W_t \sim WN(0,1). \] Then

\[ \gamma(h) = \begin{cases} 3, & h = 0, \\ 2, & h = \pm 1, \\ 1, & h = \pm 2, \\ 0, & \text{otherwise}. \end{cases} \]

Note: The \( \sin \) terms always cancel each other out since \( \gamma(h) \) is an even function.

Thus, 
\[ \gamma(w) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i wh} \]
\[ = 3 + 2 (2) \cos(2\pi w) + 2 (1) \cos(4\pi w) \]
\[ = 3 + 4 \cos(2\pi w) + 2 \cos(4\pi w), \quad -\frac{1}{4} \leq w \leq \frac{1}{4}. \]

Now see my R code for the plots.
I plotted the spectral density in R. It suggests that we should observe oscillatory behavior with period 3.

Using animating, I simulated 100 values from the process. There is clear evidence of period-3 behavior on the time series plot.