We apply the Rule-Walker equations. The matrix equation for $\hat{\Phi}_1$ and $\hat{\Phi}_2$ is

\[
\begin{pmatrix}
\hat{\Phi}_1 \\
\hat{\Phi}_2
\end{pmatrix} =
\begin{pmatrix}
\hat{\delta}(1) & \hat{\delta}(2) \\
\hat{\delta}(2) & \hat{\delta}(1)
\end{pmatrix}^{-1}
\begin{pmatrix}
\hat{\delta}(1) \\
\hat{\delta}(2)
\end{pmatrix}
\]

\[=
\begin{pmatrix}
1382.2 & 1114.4 \\
1114.4 & 1382.2
\end{pmatrix}^{-1}
\begin{pmatrix}
1114.4 \\
591.73
\end{pmatrix}
\]

\[=
\frac{1}{668.589.5}
\begin{pmatrix}
1382.2 & -1114.4 \\
-1114.4 & 1382.2
\end{pmatrix}
\begin{pmatrix}
1114.4 \\
591.73
\end{pmatrix}
\]

\[=
\begin{pmatrix}
1.318 \\
-0.634
\end{pmatrix}
\]

Also, $\hat{\delta}^2 = \hat{\delta}(1) - (\hat{\Phi}_1 \hat{\Phi}_2) \begin{pmatrix}
\hat{\delta}(1) \\
\hat{\delta}(2)
\end{pmatrix}$

\[= 1382.2 - (1.318 \cdot -0.634)
\begin{pmatrix}
1114.4 \\
591.73
\end{pmatrix}
\]

\[= 1382.2 - 1093.6 = 288.6.
\]

95% CIs: Since $(\hat{\Phi}_1) \sim N((\Phi_1), \frac{1}{n} \sigma^2 \Sigma^{-1})$, the CIs are those:

$\Phi_1: 1.318 \pm 1.96 \sqrt{\frac{288.6 \cdot 1382.2}{100 \cdot 668.589.5}} = (1.167, 1.469)$

$\Phi_2: -0.634 \pm 0.151 = (-0.785, -0.483).$
(a) Here \( \phi(z) = 1 - \phi z - \phi^2 z^2 \).

Setting \( 1 - \phi z - \phi^2 z^2 = 0 \), we find that

\[
\phi^2 z^2 + \phi z - 1 = 0 \quad \text{(by quadratic formula)}
\]

\[
z = \frac{-\phi \pm \sqrt{\phi^2 - 4 \phi^2(-1)}}{2 \phi^2} = \frac{-\phi \pm \sqrt{5} \phi}{2 \phi^2}
\]

\[
= \frac{-\phi \pm \sqrt{5} \phi}{2 \phi^2} \cdot \frac{-1 \pm \sqrt{5}}{2 \phi}.
\]

We need both of these roots to be outside the unit circle to have a causal process \( T \) we need

\[
\left| \frac{-1 + \sqrt{5}}{2 \phi} \right| > 1 \quad \text{and} \quad \left| \frac{-1 - \sqrt{5}}{2 \phi} \right| > 1 \quad \Rightarrow \quad \text{we need}
\]

\[
-1 < \frac{-1 + \sqrt{5}}{2} \quad \text{and} \quad \phi < \frac{-1 - \sqrt{5}}{2 \phi}
\]

\[
0.618 \quad \text{and} \quad \phi < -1.618
\]

\( \Rightarrow \) The process is causal iff \( \left| \phi \right| < 0.618 \).

(b) Here we find the Yule-Walker equations:

Multiply by \( X_t \):

\[
\gamma(0) - \phi \gamma(1) - \phi^2 \gamma(2) = E[X_t X_t]
\]

\( \Rightarrow \)

\[
\gamma(0) - \phi \gamma(1) - \phi^2 \gamma(2) = 0 \quad (4)
\]

Multiply by \( X_{t-1} \):

\[
\gamma(1) - \phi \gamma(2) - \phi^2 \gamma(3) = E[X_{t-1} X_t]
\]

\( \Rightarrow \)

\[
\gamma(1) - \phi \gamma(2) - \phi^2 \gamma(3) = 0 \quad (4d)
\]

This \( \{X_t\} \) causal.

Multiply by \( X_{t-2} \):

\[
\gamma(2) - \phi \gamma(3) - \phi^2 \gamma(4) = E[X_{t-2} X_t]
\]

\( \Rightarrow \)

\[
\gamma(2) - \phi \gamma(3) - \phi^2 \gamma(4) = 0 \quad (4d+2)
\]

This \( \{X_t\} \) causal.

(continued)
B7 (44) \; \hat{\delta}^2 \hat{\delta}(1) + \rho \hat{\delta}(0) - \hat{\delta}(1) = 0 \Rightarrow
\hat{\delta} = \frac{-\hat{\delta}(0) \pm \sqrt{\hat{\delta}(0)^2 + 4 \hat{\delta}(1)^2}}{2 \hat{\delta}(1)}.

Here \( \hat{\delta}(0) = 6.06 \) and \( \hat{\delta}(1) = \hat{\rho} \hat{\delta}(0) = \hat{\rho}(6.06) = 0.687(6.06) \equiv 4.163. \Rightarrow
\hat{\delta} = \frac{-6.06 \pm \sqrt{6.06^2 + 4(4.163)^2}}{2(4.163)}
\equiv \frac{-6.06 \pm 10.30}{2(4.163)} \equiv -1.96 \text{ or } 0.509. \Rightarrow
Use \( \hat{\delta} = 0.509. \)

B7 (44) \; \chi(2) = \rho \chi(1) + \chi^2 \chi(0). \; \text{Substituting this into (4), we get that}
\hat{\chi}^2 = \hat{\chi}(0) - \hat{\rho} \hat{\chi}(1) - \hat{\chi}^2 \left( \hat{\rho} \hat{\chi}(1) + \hat{\rho}^2 \hat{\chi}(0) \right)
\equiv 6.06 - (0.509)(4.163) - (0.509)^2 \left[ 0.509(4.163) + 0.509^2(6.06) \right]
\equiv 2.99

**Estimated:** \( \hat{\delta} = 0.509, \hat{\chi}^2 = 2.99 \)

Please see my R code for my comments and final best models.
First, note that \( \phi(z) = 1 - 0.5z - 0.25z^2 \).

Setting \( \phi(z) = 0 \) we get

\[
0.25z^2 + 0.5z - 1 = 0 \quad \rightarrow \\
2z^2 + 2z - 4 = 0 \quad \rightarrow \\
\frac{-2 \pm \sqrt{4 - 4(-4)(-4)}}{2} = \frac{-2 \pm \sqrt{20}}{2} \\
\approx 1.236 \text{ or } -3.236 \quad \Rightarrow \text{Causal!} \\
\text{not in unit circle}
\]

Yule–Walker Equations:

Multiply by \( X_t \):

\[
\gamma(0) - 0.5\gamma(1) - 0.25\gamma(2) = 0
\]

Multiply by \( X_{t-1} \):

\[
\gamma(1) - 0.5\gamma(0) - 0.25\gamma(1) = 0
\]

Multiply by \( X_{t-2} \):

\[
\gamma(2) - 0.5\gamma(1) - 0.25\gamma(2) = 0
\]

Multiply by \( X_{t-3} \):

\[
\gamma(3) - 0.5\gamma(2) - 0.25\gamma(3) = 0
\]

By (4), \( \rho(3) = 0.25\rho(1) + 0.5\rho(2) \).

By (4), \( \rho(2) = 0.25\rho(0) + 0.5\rho(1) = 0.25 - 0.5\rho(1) \).

By (4), \( \rho(1) = 0.25\rho(1) + 0.5\rho(0) \)

\( \Rightarrow \) 0.75\rho(1) = 0.5 \quad \Rightarrow \rho(1) = \frac{2}{3}.

Then \( \rho(2) = 0.25 + 0.5\rho(1) = 0.25 + \frac{1}{3} \left( \frac{2}{3} \right) = \frac{7}{12} \).

Then \( \rho(3) = 0.25\rho(1) + 0.5\rho(2) = 0.25 \left( \frac{2}{3} \right) + 0.5 \left( \frac{7}{12} \right) \)

\( = \frac{11}{24} \).

(We could use (4) to find \( \gamma(0) \).)
Please see my R code for my modelling process.

I found that one round of log-4 differencing seemed to yield a stationary TS.

The final model that I chose was a SARIMA $(0,0,2) \times (0,1,0)_4$ model with AIC = 147.13.

The ACF of the residuals shows only one spot where an ACF value passes the bar for significance, this just barely and at a lag of 7 quarters. The Ljung-Box p-values are all consistent with white noise, and the normal Q-Q plot looks great.

See my R code for the forecast plot.

\[(1 - B^4) \times t - 0.0384\]  
\[= (1 + 0.2398 B + 0.3616 B^2) W_t, \text{ where } W_t \sim WN(0, 0.007974).\]