\[ 1.4 \quad \gamma(s,t) = E \left[ (x_s - m_s)(x_t - m_t) \right] \]
\[ = E \left[ x_s x_t - m_s x_t + m_s m_t - x_s m_t \right] \]
\[ = E \left[ x_s x_t \right] - m_s E[x_t] + m_s m_t - E[x_s] m_t \]
\[ = E \left[ x_s x_t \right] - m_s m_t + m_s m_t - m_s m_t \]
\[ = E \left[ x_s x_t \right] - m_s m_t. \]

(1.6) a) We check stationarity by checking that the mean is free of \( t \) and that \( \gamma(s,t) \) depends only on \( |s-t| \).

Mean: \[ E[x_t] = E[\beta_0 + \beta_1 t + \omega_t] \]
\[ = \beta_0 + \beta_1 t + E[\omega_t] = \beta_0 + \beta_1 t, \]
which depends on \( t \). Thus, \( x_t \) is not stationary.

b) If \( y_t = x_t - x_{t-1} \), then
\[ y_t = (\beta_0 + \beta_1 t + \omega_t) - (\beta_0 + \beta_1 (t-1) + \omega_{t-1}) \]
(continued)
\[ \gamma(t) = \beta_2 + w_t - w_{t-1}. \]

Mean:
\[
\mathbb{E}[\gamma_t] = \mathbb{E}[\beta_2 + w_t - w_{t-1}]
\]
\[
= \beta_2 + \mathbb{E}[w_t] - \mathbb{E}[w_{t-1}] = \beta_2, \text{ free of } t.
\]

Autocorrelation:
\[ \gamma(s, t) = \text{cov}(\gamma_s, \gamma_t) \]
\[ = \text{cov}(\beta_2 + w_s - w_{s-1}, \beta_2 + w_t - w_{t-1}) \]
\[ = \text{cov}(w_s, w_t) - \text{cov}(w_s, w_{t-1}) \]
\[ - \text{cov}(w_{s-1}, w_t) + \text{cov}(w_{s-1}, w_{t-1}). \]

Note that \( \text{cov}(w_s, w_t) = \begin{cases} 1 & s = t, \\ 0 & \text{otherwise}. \end{cases} \)

Thus,
\[ \gamma(s, t) = \sigma_w^2 I(s = t) - \sigma_w^2 I(s = t-1) \]
\[ - \sigma_w^2 I(s-1 = t) + \sigma_w^2 I(s = t) \]
\[ = \begin{cases} 2 \sigma_w^2, & s - t = 0, \\ -\sigma_w^2, & s - t = \pm 1, \\ 0, & \text{otherwise}. \end{cases} \]

Since \( \mathbb{E}[\gamma_t] \) is free of \( t \) and \( \gamma(s, t) \) depends only on \( |s-t| \), \( \gamma_t \) is stationary.
\[ E[V_t] = \frac{1}{2q+1} \sum_{j=-q}^{q} E[X_{t-j}] \]

\[ = \frac{1}{2q+1} \sum_{j=-q}^{q} \left( \beta_1 \frac{q+1}{2q+1} \left( t+j \right) \right) \]

\[ = \frac{1}{2q+1} \frac{q+1}{2q+1} \sum_{j=-q}^{q} \left( \beta_1 \right) + \frac{1}{2q+1} \frac{q+1}{2q+1} \sum_{j=-q}^{q} \left( \beta_2 \right) \]

\[ = \frac{\beta_1}{2q+1} \left( 2q+1 \right) + \frac{\beta_2}{2q+1} \underbrace{\sum_{j=-q}^{q} \left( t+j \right)}_{\left( 2q+1 \right) t} \]

\[ = \sqrt{\beta_1 + \beta_2 t} \]

ACVF: \[ \gamma(s+t) = \text{Cov}(V_s, V_t) = \]

\[ \text{Cov}\left( \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t-j}, \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t-j} \right) \]

\[ = \left( \frac{1}{2q+1} \right)^2 \text{Cov}\left( \frac{q+1}{2q+1} X_{S}, \frac{q+1}{2q+1} X_{T} \right) \]

Since \( \beta_1 + \beta_2 t \) is a constant, \( \text{Cov}(w_s, w_t) = \begin{cases} \sigma_w^2, & s = t \\ 0, & \text{otherwise} \end{cases} \)

Determines \( \text{Cov}\left( \frac{q+1}{2q+1} X_{S}, \frac{q+1}{2q+1} X_{T} \right) \) is how much the windows \( s-q \) to \( s+q \) and \( t-q \) to \( t+q \) overlap. Consider a few cases:
5 = \ell + 1: \text{ Here the windows are } t+1 - \ell \text{ to } t+1 + \ell \text{ and } t - \ell \text{ to } t + \ell. \text{ There are } 2\ell \text{ matches, and } \text{cov}(w_{t+1}, w_t) = \frac{(2\ell) \sigma_w^2}{(2\ell+1)^2}.

General version:

\[ g(s, t) = \begin{cases} \frac{\sigma_w^2 (2\ell+1 - |s-t|)}{(2\ell+1)^2}, & |s-t| = 0, 1, \ldots, 2\ell, \\ 0, & \text{otherwise}. \end{cases} \]

1.7

If \(|\ell| \geq 2\), then the sets \{w_{t-\ell}, \ldots, w_t, w_{t+\ell}\} and \{w_{t-1}, w_t, w_{t+1}\} don't overlap, meaning that the covariance is 0.

|\ell| = 0: \text{cov}(x_t, x_t) = \text{cov}(w_{t-1} + 2w_t + w_{t+1}, w_{t-1} + 2w_t + w_{t+1})
\quad = \text{cov}(w_{t-1}, w_{t-1}) + 4\text{cov}(w_t, w_t) + \text{cov}(w_{t+1}, w_{t+1})
\quad = 6 \sigma_w^2.

|\ell| = 1: \text{cov}(x_t, x_{t+1}) = \text{cov}(w_{t-1} + 2w_t + w_{t+1}, w_t + 2w_{t+1} + w_{t+2})
\quad = 2\text{cov}(w_t, w_t) + 2\text{cov}(w_{t+1}, w_{t+1}) = 4 \sigma_w^2.

(continued)
\( h = 2: \) \( \text{Cov}(x_t, x_{t+1}) = \text{Cov}(w_{t+1}, w_{t+1}) = \sigma_w^2. \)

ACVF:

\[
\gamma(h) = \begin{cases} 
6 \sigma_w^2, & |h| = 0, \\
4 \sigma_w^2, & |h| = 1, \\
\sigma_w^2, & |h| = 2, \\
0, & \text{otherwise.}
\end{cases}
\]

ACF:

\[
\rho(h) = \begin{cases} 
1, & |h| = 0, \\
2/3, & |h| = 1, \\
1/6, & |h| = 2, \\
0, & \text{otherwise.}
\end{cases}
\]

Dividing \( \gamma(h) \) by \( \gamma(0) \) to get \( \rho(h) \).

Plot:

\[
\begin{array}{c}
\rho(h) \\
\hline
-3 & 3 & h
\end{array}
\]

Note that \( x_0 = 0 \), \( x_1 = d + x_0 + w_1 = d + w_1 \), \( x_2 = d + x_1 + w_2 = 2d + w_1 + w_2 \), \ldots. Thus, in general, \( x_t = d t + \sum_{k=1}^{t} w_k \).

Mean:

\[
\mu_t = E[x_t] = E\left[ dt + \sum_{k=1}^{t} w_k \right] = dt.
\]
ACVF: \( \gamma(s, t) = \text{cov}(X_s, X_t) \)

\[
= \text{cov}\left( \sum_{k=1}^{s} \xi_k^t w_k, \sum_{k=1}^{t} \xi_k^t w_k \right)
\]

\[
= \text{cov}\left( \sum_{k=1}^{s} w_k, \sum_{k=1}^{t} w_k \right) \quad \text{### number of overlapping values}
\]

\[
= \min(s, t) \sigma_w^2 \quad \text{for } s, t \text{ positive integers.}
\]

C) Since the mean function \( \mu_t = \delta t \) is not constant (unless \( \delta = 0 \)), the process \( X(t) \) is not stationary.

D) \( \rho_x(t-1, t) = \frac{\gamma(t-1, t)}{\sqrt{\delta(t-1, t+1) \delta(t+1)}} = \frac{\min(t-1, t) \sigma_w^2}{\sqrt{(t-1) \cdot t \cdot \sigma_w^2}} \)

\[
= \frac{t-1}{\sqrt{(t-1) \cdot t}} = \sqrt{\frac{t-1}{t}}, \quad \text{which goes to 1 as } t \to \infty.
\]

This result implies that \( X(t) \) is not stationary. It also implies that values at adjacent times are more and more highly correlated as the time increases.

E) Use differencing. \( \nabla X_t = X_t - X_{t-1} \)

\[
= (\delta t + \sum_{k=1}^{t-1} w_k) - (\delta(t-1) + \sum_{k=1}^{t-1} w_k) = \delta + w_t.
\]
Since the process \( \{ w_t \} \) is stationary and \( \delta \) is a constant, the sum \( \{ \delta + w_t \} \) is also stationary. The mean function is \( \mu_t = E [ \delta + w_t ] = \delta \) and the ACF is
\[
\gamma(s, t) = \begin{cases} \sigma^2 & s = t, \\ 0 & \text{otherwise}. \end{cases}
\]